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# Soliton solutions for the positive Gardner-KP equation by $(G'/G, 1/G)$ – Expansion method

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**Abstract** The $(G'/G, 1/G)$ – expansion method is one of the most direct and effective method for obtaining abundant new traveling wave solutions of nonlinear partial differential equations. In this article, we construct exact soliton solutions of nonlinear evolution equation via positive Gardner-KP equation by using $(G'/G, 1/G)$ – expansion method, where $G(\xi)$ satisfies the auxiliary ordinary differential equation (ODE) $G''(\xi) + \lambda G(\xi) = \mu$; $\lambda$ and $\mu$ are arbitrary constants. The three type of the solutions are found as, hyperbolic, trigonometric and rational function form involving more parameters and some of our constructed solutions are identical with results obtained by other authors if certain parameters take special values.



## 1. Introduction

Nonlinear evolution equations play a significant role in various scientific and engineering fields, such as, optical fibers, solid state physics, fluid mechanics, plasma physics, chemical kinematics, the heat flow and the wave propagation phenomena, quantum mechanics, propagation of shallow water waves etc. Nonlinear wave phenomena of diffusion, reaction, dispersion, dissipation, and convection are very important in nonlinear wave equations. In recent years, the exact solutions of nonlinear partial differential equations have been investigated by many researchers (see for example [1–32]) who are concerned in nonlinear physical phenomena and many powerful and efficient methods have been offered by them. Among non-integrable nonlinear differential equations there is a wide class of equations that referred to as the partially integrable, because these equations become integrable for some values of their parameters. There are many different methods to look for the exact solutions of these equations. The most famous algorithms are truncated Painleve expansion method [1], Tanh-function method [2–4], Jacobi elliptic function expansion method [5–7], Variational iteration method [8,10], Inverse scattering transform method [11], Hirota method [12], Truncated Painleve expansion method [13], Backlund transform method [14] and Exp-function method [15,16] are used for searching the exact solutions.

Wang et al. [17] introduced a direct and concise method, called $(G'/G)$ – expansion method to look for traveling wave solutions of nonlinear partial differential equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$; $\lambda$ and $\mu$ are arbitrary constants. For additional references see the articles [18–23]. It is to be highlighted that Li et al. [24] applied $(G'/G, 1/G)$ – expansion method for finding exact traveling wave solutions of nonlinear evolution equations, and hence established it as

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an extension of the recently proposed $(G'/G)$ – expansion method. They calculated abundant traveling wave solutions with arbitrary parameters of the Zakharov equations. Moreover, they also rediscovered the well-known solitary wave solutions when the parameters are replaced by special values. It is worth mentioning that, recently, Zayed et al. [25,26] also applied $(G'/G, 1/G))$ – expansion method for finding traveling wave solutions of the nonlinear (3 + 1)-dimensional Kadomtsev–Petviashvili, nonlinear KdV–mKdV equations. For detailed study on $(G'/G)$ – expansion method, the readers are referred to [27–29]. Inspired and motivated by the ongoing research in this area, we extended the approach given in [24–26] which is called the $(G'/G, 1/G)$ – expansion method for finding the exact traveling wave solutions of positive Gardner-KP equation. Moreover, it may be concluded that some very useful, new traveling wave solutions of the nonlinear PDEs can be obtained by making an appropriate use of the presented scheme $((G'/G, 1/G)$ – expansion method.

## 2. Methodology

In this section, we describe the main steps of the $(G'/G, 1/G)$ – expansion method [24–26] for finding traveling wave solutions of nonlinear evolution equation. Suppose a nonlinear equation for $P(x, y, t)$ is given by

$$P(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0, \tag{1}$$

in which both nonlinear term(s) and higher order derivatives of $P(x, y, t)$ are all involved. In general, the left-hand side of Eq. (1) is a polynomial in $\psi$ and its various derivatives. The $(G'/G, 1/G)$ – expansion method for solving Eq. (1) proceeds in the following steps:

*Step 1:* Look for traveling wave solution of Eq. (1) by taking

$$P = P(\xi), \quad \xi = x + y - Vt, \tag{2}$$

where $V$ is nonzero constant, $P(\xi)$ the function of $\xi$. Substituting (2) into Eq. (1) yields an ordinary differential equation (ODE) for $P(\xi)$.

$$Q(u, u', u'', u''', \ldots) = 0. \tag{3}$$

*Step 2:* If possible, integrate Eq. (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) may be set to zero.

*Step 3:* According to the $(G'/G, 1/G)$ – expansion method, suppose that the solution of Eq. (3) can be expressed by a finite power series in $\varphi$ and $\psi$ as follows:

$$u(\xi) = \sum_{n=0}^{M} a_n \varphi^n + \sum_{n=1}^{M} b_n \varphi^{n-1} \psi, \tag{4}$$

where $a_n$ $(n = 1, 2, 3, \ldots, M)$ and $b_n$ $(n = 1, 2, 3, \ldots, M)$ are constants to be determine later and $\varphi(\xi)$ and $\psi(\xi)$ are given by

$$\varphi(\xi) = \left(\frac{G'(\xi)}{G(\xi)}\right), \quad \psi(\xi) = \left(\frac{1}{G(\xi)}\right), \tag{5}$$

which satisfied

$$G''(\xi) + \lambda G(\xi) = \mu. \tag{6}$$

Eqs. (5) and (6) yields

$$\varphi' = -\varphi^2 + \mu\varphi - \lambda, \quad \psi' = -\varphi\psi. \tag{7}$$

From the three cases of general solutions of the Eq. (6), we have:

*Case 1:* When $\lambda < 0$ the general solution of Eq. (6) is

$$G(\xi) = A\sinh(\sqrt{-\lambda}\xi) + B\cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda},$$

we have

$$\psi^2 = -\frac{\lambda}{\lambda^2\sigma + \mu^2}(\varphi^2 - 2\mu\psi + \lambda), \tag{8}$$

where $A$ and $B$ are two arbitrary constants and $\sigma = A^2 - B^2$.

*Case 2:* When $\lambda > 0$ the general solution of Eq. (6) is

$$G(\xi) = A\sin(\sqrt{\lambda}\xi) + B\cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda},$$

we have

$$\psi^2 = \frac{\lambda}{\lambda^2\varepsilon - \mu^2}(\varphi^2 - 2\mu\psi + \lambda), \tag{9}$$

where $A$ and $B$ are two arbitrary constants and $\varepsilon = A^2 + B^2$.

*Case 3:* When $\lambda = 0$ the general solution of Eq. (6) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A\xi + B,$$

and we have

$$\psi^2 = \frac{1}{A^2 - 2\mu B}(\varphi^2 - 2\mu\psi), \tag{10}$$

where $A$ and $B$ are two arbitrary constants.

*Step 4:* Determine $M$. This, usually, can be accomplished by balancing the linear term of highest order with the highest order nonlinear term obtained in Step 2.

*Step 5:* Substituting Eq. (4) into Eq. (3), using Eqs. (7) and (8) will yield a polynomial in $\varphi$ and $\psi$ in which the degree of $\psi$ is not larger than 1. Compare the like powers of $\varphi^M$ and $\varphi^M\psi$ equal to zero, yields a set of algebraic equations for $a_n$ $(n = 0, 1, 2, 3, \ldots, M)$ and $b_n$ $(n = 1, 2, 3, \ldots, M), \mu, \lambda, A, B$ and $V$.

*Step 6:* Solve the system which is obtained in step 5 for $a_n$ $(n = 0, 1, 2, 3, \ldots, M)$ and $b_n$ $(n = 1, 2, 3, \ldots, M), \mu, \lambda, A, B$ and $V$ with the help of Maple, to determine these constants. Putting the values of these constants into Eq. (4), one can obtain the traveling wave solutions expressed by the hyperbolic functions of Eq. (2). We can obtain the more general type and new exact traveling wave solution of the nonlinear partial differential Eq. (1).

*Step 7:* Similarly substituting Eq. (4) into Eq. (3), using Eqs. (7) and (9) (or Eqs. (7) and (10)) will yield a polynomial in $\varphi$ and $\psi$ in which the degree of $\psi$ is not larger than 1.Compare the like powers of $\varphi^M$ and $\varphi^M\psi$ equal to zero, yields a set of algebraic equations for $a_n$ $(n = 0, 1, 2, 3, \ldots, M)$ and $b_n$ $(n = 1, 2, 3, \ldots, M), \mu, \lambda, A, B$ and $V$, we obtain traveling wave solutions of Eq. (1) which are expressed by trigonometric functions (or expressed by rational functions) as proceeding before.

## 3. Application to positive Gardner-KP equation

Let us consider the positive Gardner-KP equation as follows

$$(u_t + 6uu_x + 6u^2u_x + u_{xxx})_x + u_{yy} = 0. \tag{11}$$

Shafiof et al. [30] used $(G'/G)$ – expansion method to find the traveling wave solutions of the positive and negative Gardner-KP equation. Wazwaz [31] applied Hirota's bilinear method to obtain multiple-soliton and multiple singular soliton solutions for the Gardner-KP equation. Bin and Qiang [32] obtained the symmetries and group invariant solutions to the Gardner-KP equation by using the direct symmetry method.

To solve the Eq. (11), using the wave transformation Eq. (2) into Eq. (11), and integrating once and setting the constant of integration to zero, we have

$$(-V+1)u' + 6uu' + 6u^2u' + u''' = 0, \tag{12}$$

Applying the balancing principle between $u'''$ and $u^2u'$ in Eq. (12), we get $M = 1$. Therefore the trail solution is

$$u = a_0 + a_1\varphi(\xi) + b_1\psi(\xi), \tag{13}$$

where $a_0$, $a_1$ and $b_1$ are constants to be determined later. Here, we discuss three cases as follows.

*Case 1:* When $\lambda < 0$ (Hyperbolic function solutions).

If $\lambda < 0$, substituting Eq. (13) into Eq. (12) and using Eqs. (7) and (8), the left-hand side of Eq. (12) becomes a polynomial in $\varphi$and$\psi$. Setting the coefficients of this polynomial to zero yields a system of algebraic equations in $a_0$, $a_1, b_1, \mu, \lambda$, $V$ and $\sigma$ as follows:

$$\varphi^4 : \frac{18a_1b_1^2\lambda}{\lambda^2\sigma+\mu^2} - 6a_1^3 - 6a_1 = 0,$$

$$\varphi^3 : \frac{12a_0b_1^2\lambda}{\lambda^2\sigma+\mu^2} - \frac{12a_1^2b_1\lambda\mu}{\lambda^2\sigma+\mu^2} - \frac{6b_1\lambda\mu}{\lambda^2\sigma+\mu^2} + \frac{12a_0b_1^2\lambda}{\lambda^2\sigma+\mu^2} + \frac{12b_1^3\lambda^2\mu}{(\lambda^2\sigma+\mu^2)^2} - 12a_0a_1^2 - 6a_1^2 = 0,$$

$$\varphi^2 : \frac{24a_1b_1^2\lambda^2}{\lambda^2\sigma+\mu^2} + \frac{3a_1\lambda\mu^2}{\lambda^2\sigma+\mu^2} - \frac{12a_0a_1b_1\lambda\mu}{\lambda^2\sigma+\mu^2} - \frac{6a_1b_1\lambda\mu}{\lambda^2\sigma+\mu^2} - \frac{12a_1b_1^2\lambda^2\mu^2}{(\lambda^2\sigma+\mu^2)^2} - 6a_0^2a_1 - 6a_1^3\lambda - 8a_1\lambda - 6a_0a_1 - a_1 + Va_1 = 0,$$

$$\varphi^1 : \frac{12a_0b_1^2\lambda^2}{\lambda^2\sigma+\mu^2} - \frac{6b_1\lambda^2\mu}{\lambda^2\sigma+\mu^2} - \frac{12a_1^2b_1\lambda^2\mu}{\lambda^2\sigma+\mu^2} + \frac{6b_1^2\lambda^2}{\lambda^2\sigma+\mu^2} + \frac{12b_1^3\lambda^3\mu}{(\lambda^2\sigma+\mu^2)^2} - 12a_0a_1^2\lambda - 6a_1^2\lambda - = 0,$$

$$\varphi^3\psi : \frac{6b_1^3\lambda}{\lambda^2\sigma+\mu^2} - 18a_1^2b_1 - 6b_1 - = 0,$$

$$\varphi^2\psi : \frac{42a_1b_1^2\lambda\mu}{\lambda^2\sigma+\mu^2} + 24a_0a_1b_1 - 12a_1\mu - 6a_1^3\mu + 12a_1b_1 = 0,$$

$$\varphi\psi : \frac{24a_1^2b_1\lambda\mu^2}{\lambda^2\sigma+\mu^2} - \frac{24a_0b_1^2\lambda\mu}{\lambda^2\sigma+\mu^2} + \frac{12b_1\lambda\mu^2}{\lambda^2\sigma+\mu^2} + \frac{6b_1^3\lambda^2}{\lambda^2\sigma+\mu^2} - \frac{12b_1^2\lambda\mu}{\lambda^2\sigma+\mu^2} - \frac{24b_1^3\lambda^2\mu^2}{(\lambda^2\sigma+\mu^2)^2} + 12a_0a_1^2\mu - 12a_1^2b_1\lambda - 5b_1\lambda - 6a_0^2b_1 + 6a_1^2\mu - 6a_0b_1 - b_1 + Vb_1 = 0,$$

$$\psi^1 : \frac{24a_0a_1b_1\lambda\mu^2}{\lambda^2\sigma+\mu^2} - \frac{18a_1b_1^2\lambda^2\mu}{\lambda^2\sigma+\mu^2} - \frac{6a_1\lambda\mu^3}{\lambda^2\sigma+\mu^2} + \frac{12a_1b_1\lambda\mu^2}{\lambda^2\sigma+\mu^2} + \frac{24a_1b_1^2\lambda^2\mu^3}{(\lambda^2\sigma+\mu^2)^2} + 6a_0^2a_1\mu - 12a_0a_1b_1\lambda + 5a_1\lambda\mu - 6a_1b_1\lambda + 6a_0a_1\mu + a_1\mu - Va_1\mu = 0,$$

$$\psi^0 : \frac{3a_1\lambda^2\mu^2}{\lambda^2\sigma+\mu^2} + \frac{6a_1b_1^2\lambda^3}{\lambda^2\sigma+\mu^2} - \frac{12a_0a_1b_1\lambda^2\mu}{\lambda^2\sigma+\mu^2} - \frac{6a_1b_1\lambda^2\mu}{\lambda^2\sigma+\mu^2} - \frac{12a_1b_1^2\lambda^3\mu^2}{(\lambda^2\sigma+\mu^2)^2} - 6a_0^2a_1\lambda - 2a_1\lambda^2 - 6a_0a_1\lambda - a_1\lambda + Va_1\lambda = 0. \tag{14}$$

Solving the above system, we have the following solution sets:

Result 1. We have

$$a_0 = -\frac{1}{2}\left(1 + \frac{\mu\sqrt{\lambda}}{\sqrt{\lambda^2\sigma+\mu^2}}\right), a_1 = 0,$$
$$b_1 = \sqrt{\frac{(\lambda^2\sigma+\mu^2)}{\lambda}}, V = -\frac{2\lambda^3\sigma+\lambda^2\sigma-\lambda\mu^2+\mu^2)}{2(\lambda^2\sigma+\mu^2)},$$
$$\sigma = A^2 - B^2. \tag{15}$$

Now, the traveling wave solution of Eq. (11) become

$$u(x,y,t) = -\frac{1}{2}\left(1 + \frac{\mu\sqrt{\lambda}}{\sqrt{\lambda^2\sigma+\mu^2}}\right) + \sqrt{\frac{(\lambda^2\sigma+\mu^2)}{\lambda}} \times \left(\frac{\lambda}{A\sinh(\sqrt{-\lambda}\xi)\lambda + B\cosh(\sqrt{-\lambda}\xi)\lambda + \mu}\right), \tag{16}$$

$$\text{where} \quad \xi = x + y + \left(\frac{2\lambda^3\sigma+\lambda^2\sigma-\lambda\mu^2+\mu^2)}{2(\lambda^2\sigma+\mu^2)}\right)t. \tag{17}$$

In particular, if we take $A = 0, B > 0$ and $\mu = 0$ in Eq. (16), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + i\sqrt{\lambda}\sec h(\sqrt{-\lambda}\xi), \tag{18}$$

and, if we set $A > 0, B = 0$ and $\mu = 0$ in Eq. (16), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + \sqrt{\lambda}\csc h(\sqrt{-\lambda}\xi). \tag{19}$$

Result 2. We have

$$a_0 = -\frac{1}{2}, \quad a_1 = \frac{1}{2}i, \quad b_1 = \sqrt{\frac{\lambda^2\sigma+\mu^2}{4\lambda}},$$
$$V = \frac{\lambda-1}{2}, \quad \sigma = A^2 - B^2. \tag{20}$$

Now, the traveling wave solution of Eq. (11) becomes:

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}i \times \left(\frac{(-\lambda)^{3/2}(A\cosh(\sqrt{-\lambda}\xi) + B\sinh(\sqrt{-\lambda}\xi))}{A\sinh(\sqrt{-\lambda}\xi)\lambda + B\cosh(\sqrt{-\lambda}\xi)\lambda + \mu}\right) + \sqrt{\frac{\lambda^2\sigma+\mu^2}{4\lambda}} \times \left(\frac{\lambda}{A\sinh(\sqrt{-\lambda}\xi)\lambda + B\cosh(\sqrt{-\lambda}\xi)\lambda + \mu}\right), \tag{21}$$

$$\text{where} \quad \xi = x + y - \left(\frac{\lambda-1}{2}\right)t. \tag{22}$$

In particular, by setting $A = 0, B > 0$ and $\mu = 0$ in Eq. (21), we have the following solitary solution

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}\sqrt{\lambda}\tanh\left(\sqrt{-\lambda}\xi\right) + \frac{1}{2}i\sqrt{\lambda}\sec h(\sqrt{-\lambda}\xi), \tag{23}$$

while, if we set $A > 0, B = 0$ and $\mu = 0$ in Eq. (21), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}\sqrt{\lambda}\left(\coth(\sqrt{-\lambda}\xi) + \csc h(\sqrt{-\lambda}\xi)\right). \tag{24}$$

**Case 2**: When $\lambda > 0$ (Trigonometric function solutions).

If $\lambda > 0$, substituting Eq. (13) into Eq. (12) and using Eqs. (7) and (9), the left-hand side of Eq. (12) becomes a polynomial in $\varphi$ and $\psi$. Setting the coefficients of this polynomial to zero yields a system of algebraic equations in $a_0$, $a_1, b_1, \mu, \lambda$, $V$ and $\varepsilon$ as follows:

$$\varphi^4: \frac{18a_1b_1^2\lambda}{\lambda^2\varepsilon - \mu^2} + 6a_1^3 + 6a_1 = 0,$$

$$\varphi^3: \frac{12a_1^2b_1\lambda\mu}{\lambda^2\varepsilon - \mu^2} - \frac{12a_0b_1^2\lambda}{\lambda^2\varepsilon - \mu^2} + \frac{6b_1\lambda\mu}{\lambda^2\varepsilon - \mu^2} - \frac{6b_1^2\lambda}{\lambda^2\varepsilon - \mu^2} + \frac{12b_1^3\lambda^2\mu}{(\lambda^2\varepsilon - \mu^2)^2} - 12a_0a_1^2 - 6a_1^2 = 0,$$

$$\varphi^2: \frac{6a_1b_1\lambda\mu}{\lambda^2\varepsilon - \mu^2} - \frac{24a_1b_1^2\lambda^2}{\lambda^2\varepsilon - \mu^2} - \frac{3a_1\lambda\mu^2}{\lambda^2\varepsilon - \mu^2} + \frac{12a_0a_1b_1\lambda\mu}{\lambda^2\varepsilon - \mu^2} - \frac{12a_1b_1^2\lambda^2\mu^2}{(\lambda^2\varepsilon - \mu^2)^2} - 6a_0^2a_1 - 6a_1^3\lambda - 8a_1\lambda - 6a_0a_1 - a_1 + Va_1 = 0,$$

$$\varphi^1: \frac{12a_1^2b_1\lambda^2\mu}{\lambda^2\varepsilon - \mu^2} - \frac{12a_0b_1^2\lambda^2}{\lambda^2\varepsilon - \mu^2} + \frac{6b_1\lambda^2\mu}{\lambda^2\varepsilon - \mu^2} - \frac{6b_1^2\lambda^2}{\lambda^2\varepsilon - \mu^2} + \frac{12b_1^3\lambda^3\mu}{(\lambda^2\varepsilon - \mu^2)^2} - 12a_0a_1^2\lambda - 6a_1^2\lambda - = 0,$$

$$\varphi^3\psi: \frac{6b_1^3\lambda}{\lambda^2\varepsilon-\mu^2} + 18a_1^2b_1 + 6b_1 - = 0,$$

$$\varphi^2\psi: \frac{42a_1b_1^2\lambda\mu}{\lambda^2\varepsilon-\mu^2} - 24a_0a_1b_1 + 12a_1\mu + 6a_1^3\mu - 12a_1b_1 = 0,$$

$$\varphi\psi: \frac{24a_0b_1^2\lambda\mu}{\lambda^2\varepsilon-\mu^2} - \frac{24a_1^2b_1\lambda\mu^2}{\lambda^2\varepsilon-\mu^2} - \frac{12b_1\lambda\mu^2}{\lambda^2\varepsilon-\mu^2} - \frac{6b_1^3\lambda^2}{\lambda^2\varepsilon-\mu^2} + \frac{12b_1^2\lambda\mu}{\lambda^2\varepsilon-\mu^2} - \frac{24b_1^3\lambda^2\mu^2}{(\lambda^2\varepsilon-\mu^2)^2} + 12a_0a_1^2\mu - 12a_1^2b_1\lambda - 5b_1\lambda - 6a_0^2b_1 + 6a_1^2\mu - 6a_0b_1 - b_1 + Vb_1 = 0,$$

$$\psi^1: \frac{6a_1\lambda\mu^3}{\lambda^2\varepsilon-\mu^2} - \frac{24a_0a_1b_1\lambda\mu^2}{\lambda^2\varepsilon-\mu^2} + \frac{18a_1b_1^2\lambda^2\mu}{\lambda^2\varepsilon-\mu^2} - \frac{12a_1b_1\lambda\mu^2}{\lambda^2\varepsilon-\mu^2} + \frac{24a_1b_1^2\lambda^2\mu^3}{(\lambda^2\varepsilon-\mu^2)^2} + 6a_0^2a_1\mu - 12a_0a_1b_1\lambda + 5a_1\lambda\mu - 6a_1b_1\lambda + 6a_0a_1\mu + a_1\mu - Va_1\mu = 0,$$

$$\psi^0: \frac{12a_0a_1b_1\lambda^2\mu}{\lambda^2\varepsilon-\mu^2} - \frac{3a_1\lambda^2\mu^2}{\lambda^2\varepsilon-\mu^2} - \frac{6a_1b_1^2\lambda^3}{\lambda^2\varepsilon-\mu^2} + \frac{6a_1b_1\lambda^2\mu}{\lambda^2\varepsilon-\mu^2} - \frac{12a_1b_1^2\lambda^3\mu^2}{(\lambda^2\varepsilon-\mu^2)^2} - 6a_0^2a_1\lambda - 2a_1\lambda^2 - 6a_0a_1\lambda - a_1\lambda + Va_1\lambda = 0. \tag{25}$$

Solving the above system of algebraic equations, we obtain the following results:

Result 1. We have

$$a_0 = -\frac{1}{2}\left(1 + \frac{\mu\sqrt{\lambda}}{\sqrt{\mu^2 - \lambda^2\varepsilon}}\right), a_1 = 0, b_1 = \sqrt{\frac{\mu^2 - \lambda^2\varepsilon}{\lambda}},$$
$$V = -\frac{2\lambda^3\varepsilon + \lambda^2\varepsilon + \lambda\mu^2 - \mu^2)}{2(\lambda^2\varepsilon - \mu^2)}, \varepsilon = A^2 + B^2. \tag{26}$$

Now, the traveling wave solution of Eq. (11) becomes:

$$u(x,y,t) = -\frac{1}{2}\left(1 + \frac{\mu\sqrt{\lambda}}{\sqrt{\mu^2 - \lambda^2\varepsilon}}\right) + \sqrt{\frac{\mu^2 - \lambda^2\varepsilon}{\lambda}} \times \left(\frac{\lambda}{A\sin(\sqrt{\lambda}\xi)\lambda + B\cos(\sqrt{\lambda}\xi)\lambda + \mu}\right), \tag{27}$$

$$\text{where} \quad \xi = x + y + \left(\frac{2\lambda^3\varepsilon + \lambda^2\varepsilon + \lambda\mu^2 - \mu^2}{2(\lambda^2\varepsilon - \mu^2)}\right)t. \tag{28}$$

In particular, if we take $A > 0, B = 0$ and $\mu = 0$ in Eq. (27), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + i\sqrt{\lambda}\csc(\sqrt{\lambda}\xi), \tag{29}$$

while, if we set $A = 0, B > 0$ and $\mu = 0$ in Eq. (27), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + i\sqrt{\lambda}\sec(\sqrt{\lambda}\xi). \tag{30}$$

Result 2. We have

$$a_0 = -\frac{1}{2}, \; a_1 = \frac{1}{2}i, \; b_1 = \sqrt{\frac{\mu^2 - \lambda^2\varepsilon}{4\lambda}}, \; V = \frac{\lambda - 1}{2},$$
$$\varepsilon = A^2 + B^2. \tag{31}$$

Now, in this result the traveling wave solution of Eq. (11) becomes:

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}i \times \left(\frac{(\lambda)^{3/2}(A\cos(\sqrt{\lambda}\xi) - B\sin(\sqrt{\lambda}\xi))}{A\sin(\sqrt{\lambda}\xi)\lambda + B\cos(\sqrt{\lambda}\xi)\lambda + \mu}\right) + \sqrt{\frac{\mu^2 - \lambda^2\varepsilon}{4\lambda}} \times \left(\frac{\lambda}{A\sin(\sqrt{\lambda}\xi)\lambda + B\cos(\sqrt{\lambda}\xi)\lambda + \mu}\right), \tag{32}$$

$$\text{where} \quad \xi = x + y - \left(\frac{\lambda - 1}{2}\right)t. \tag{33}$$

In particular, by setting $A = 0, B > 0$ and $\mu = 0$ in Eq. (32), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}i\sqrt{\lambda}(-\tan(\sqrt{\lambda}\xi) + \sec(\sqrt{\lambda}\xi)), \tag{34}$$

while, if we set $A > 0, B = 0$ and $\mu = 0$ in Eq. (32), we have the solitary solution

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}i\sqrt{\lambda}(\cot(\sqrt{\lambda}\xi) + \csc(\sqrt{\lambda}\xi)). \tag{35}$$

*Case 3:* When $\lambda = 0$ (Rational function solutions).

If $\lambda = 0$, substituting Eq. (13) into Eq. (12) and using Eqs. (7) and (10), the left-hand side of Eq. (12) becomes a polynomial in $\varphi$ and $\psi$. Setting the coefficients of this polynomial to zero yields a system of algebraic equations in $a_0$, $a_1$, $b_1$, $\mu$ and $V$ as follows:

$$\varphi^4: \frac{18a_1b_1^2}{A^2 - 2\mu B} + 6a_1^3 + 6a_1 = 0,$$

$$\varphi^3:\ \frac{6b_1\mu}{A^2-2\mu B}-\frac{6b_1^2}{A^2-2\mu B}-\frac{12a_0b_1^2}{A^2-2\mu B}+\frac{12a_1^2b_1\mu}{A^2-2\mu B}+\frac{12b_1^3\mu}{(A^2-2\mu B)^2}-12a_0a_1^2-6a_1^2=0,$$

$$\varphi^2:\ \frac{12a_0a_1b_1\mu}{A^2-2\mu B}-\frac{3a_1\mu^2}{A^2-2\mu B}+\frac{6a_1b_1\mu}{A^2-2\mu B}-\frac{12a_1b_1^2\mu^2}{(A^2-2\mu B)^2}-6a_0^2a_1-6a_0a_1-a_1+Va_1=0,$$

$$\varphi^3\psi:\ \frac{6b_1^3}{A^2-2\mu B}+18a_1^2b_1+6b_1=0,$$

$$\varphi^2\psi:\ \frac{42a_1b_1^2\mu}{A^2-2\mu B}-24a_0a_1b_1+12a_1\mu+6a_1^3\mu-12a_1b_1=0,$$

$$\varphi\psi:\ \frac{24a_0b_1^2\mu}{A^2-2\mu B}-\frac{12b_1\mu^2}{A^2-2\mu B}+\frac{12b_1^2\mu}{A^2-2\mu B}-\frac{24a_1^2b_1\mu^2}{A^2-2\mu B}-\frac{24b_1^3\mu^2}{(A^2-2\mu B)^2}+12a_0a_1^2\mu-6a_0^2b_1+6a_1^2\mu-6a_0b_1-b_1+Vb_1=0,$$

$$\psi^1:\ \frac{6a_1\mu^3}{A^2-2\mu B}-\frac{24a_0a_1b_1\mu^2}{A^2-2\mu B}-\frac{12a_1b_1\mu^2}{A^2-2\mu B}+\frac{24a_1b_1^2\mu^3}{(A^2-2\mu B)^2}+6a_0^2a_1\mu+6a_0a_1\mu+a_1\mu-Va_1\mu=0. \quad (36)$$

Solving the above system of algebraic equations, we obtain the following results.Result 1. We have

$$a_0=-\frac{1}{2}\left(1+\frac{\mu}{\sqrt{-(A^2-2\mu B)}}\right),\ a_1=0,$$
$$b_1=\sqrt{-(A^2-2\mu B)},\ V=-\frac{A^2-2\mu B+3\mu^2}{2(A^2-2\mu B)}. \quad (37)$$

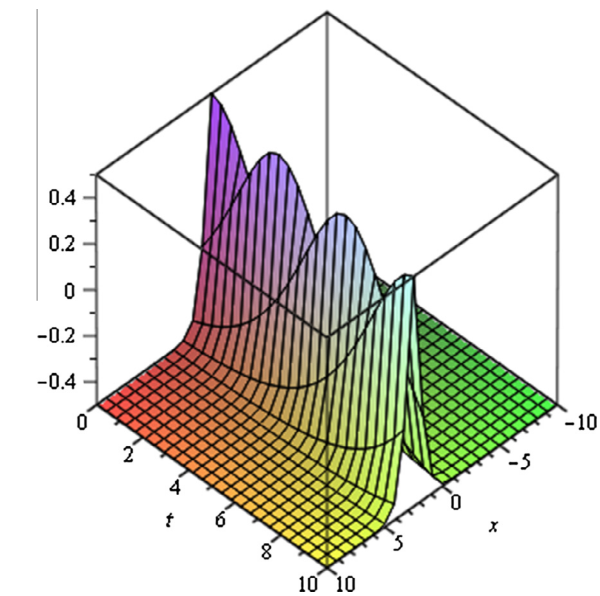

**Figure 1** Soliton solution for Eq. (18) for $\lambda=-1$.

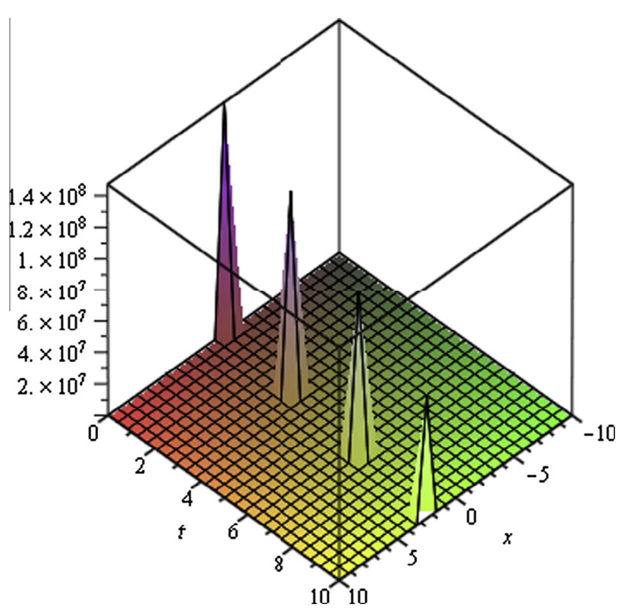

**Figure 2** Soliton solution for Eq. (19) for $\lambda=-1$.

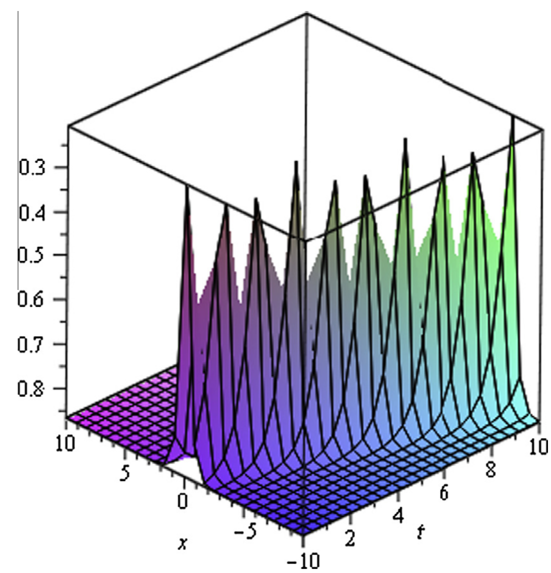

**Figure 3** Soliton solution for Eq. (23) for $\lambda=-1$.

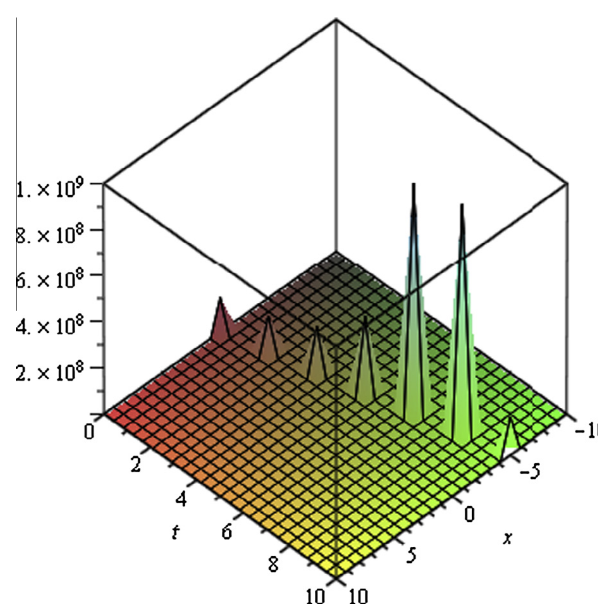

**Figure 4** Soliton solution for Eq. (24) for $\lambda=-1$.

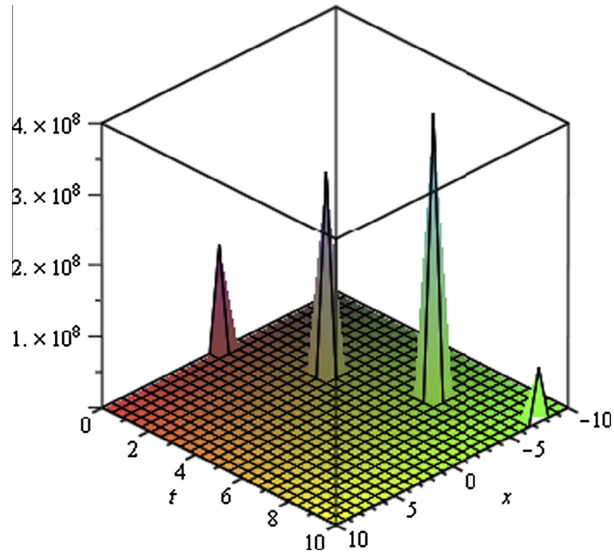


**Figure 5** Soliton solution for Eq. (29) for $\lambda = 1$.

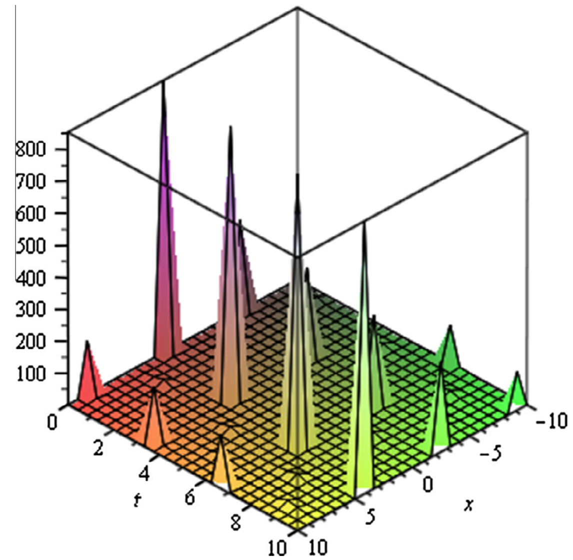


**Figure 7** Soliton solution for Eq. (34) for $\lambda = 2$.

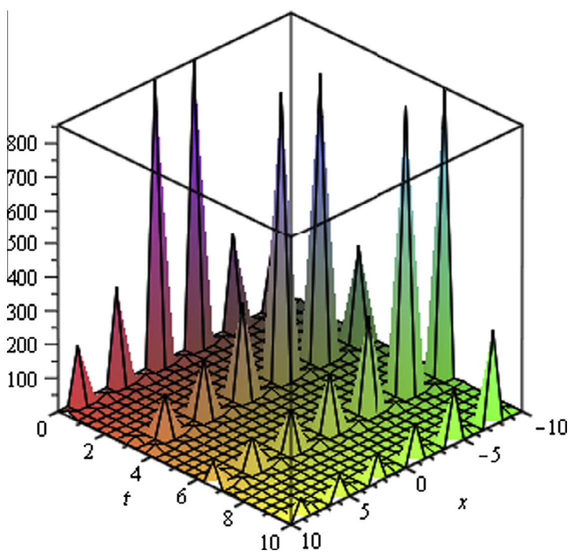


**Figure 6** Soliton solution for Eq. (30) for $\lambda = 2$.

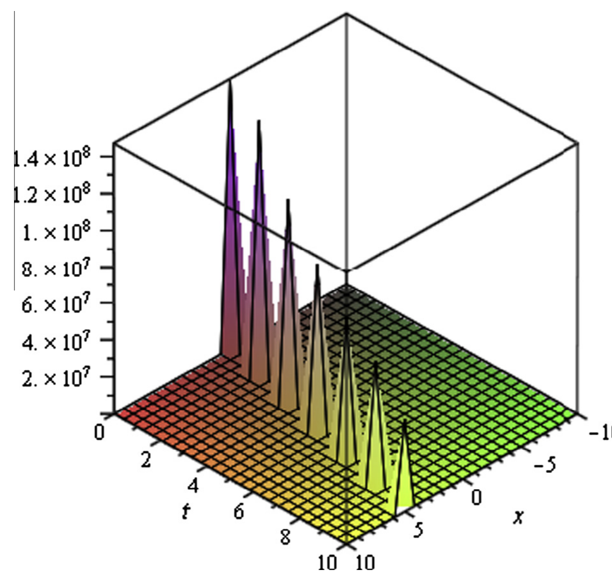


**Figure 8** Soliton solution for Eq. (35) for $\lambda = 2$.

Now, the traveling wave solution of Eq. (11) becomes:

$$u(x,y,t) = a_0 = -\frac{1}{2}\left(1 + \frac{\mu}{\sqrt{-(A^2 - 2\mu B)}}\right. \\ \left. + \sqrt{-(A^2 - 2\mu B)} \times \left(\frac{1}{(\mu/2)\xi^2 + A\xi + B}\right)\right), \tag{38}$$

where $\xi = x + y + \left(\dfrac{A^2 - 2\mu B + 3\mu^2}{2(A^2 - 2\mu B)}\right)t.$ (39)

Result 2. We have

$$a_0 = -\frac{1}{2},\ a_1 = \frac{1}{2}i,\ b_1 = \frac{1}{2}\sqrt{-(A^2 - 2\mu B)},\ V = -\frac{1}{2}. \tag{40}$$

Now, in this result the traveling wave solution of Eq. (11) becomes:

$$u(x,y,t) = -\frac{1}{2} + \frac{1}{2}i \times \left(\frac{A + \mu\xi}{(\mu/2)\xi^2 + A\xi + B}\right) \\ + \frac{1}{2}\sqrt{-(A^2 - 2\mu B)} \times \left(\frac{1}{(\mu/2)\xi^2 + A\xi + B}\right), \tag{41}$$

where $\xi = x + y + \dfrac{1}{2}t.$ (42)

*Remark:* All the solutions have been checked with Maple by putting them back into the original equation and found correct.

## 4. Graphical presentation

Graph is a powerful tool for communication that describes lucidly the solutions of the problems. Therefore, some graphs of the solutions are given (Figs. 1–8). The graphs readily have shown the solitary wave form of the solutions.

## 5. Conclusions

In this article, the $(G'/G, 1/G)$ – expansion method is successfully implemented to investigate the nonlinear partial differential equation, namely, positive Gardner-KP equation. We have constructed abundant exact traveling wave solutions including, hyperbolic function, trigonometric function and rational function solutions. The method used in this article is more effective and general than the original $(G'/G)$ – expansion method. The main advantage of this method over other methods is that, it possesses all the three types of the solutions. Therefore, this simple and powerful method can be more successfully applied to study nonlinear partial differential equations which frequently arise in engineering sciences, mathematical physics and other scientific fields.

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